Gauss-Jordan Elimination and Basic Matrix Operations

Finite Math

25 October 2018

Quiz

Give the augmented matrix for the system

$$\begin{array}{rcrcrcl} 2x & - & y & - & 3z & = & 8 \\ x & - & 2y & & & = & 7 \end{array}$$

Non-Square Systems

Example

Solve by Gauss-Jordan elimination:

$$\begin{array}{rcrcrcl} 2x & - & y & - & 3z & = & 8 \\ x & - & 2y & & & = & 7 \end{array}$$

Application

Example

A company that rents small moving trucks wants to purchase 16 *trucks with a combined capacity of* 19,200 *cubic feet. Three different types of trucks are available: a cargo van with a capacity of* 300 *cubic feet, a* 15*-foot truck with a capacity of* 900 *cubic feet, and a* 24*-foot truck with a capacity of* 1,500*-cubic feet. How many of each type should the company purchase?*

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is true if and only if

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Addition and Subtraction

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Example

Find the indicated operations

(a)

$$\begin{bmatrix} 3 & 2 \\ -1 & -1 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} -2 & 3 \\ 1 & -1 \\ 2 & -2 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 3 & 2 \\ 5 & 0 \end{bmatrix} - \begin{bmatrix} 2 & -2 \\ 3 & 4 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 3 & 2 & -1 \\ -1 & 0 & 3 \end{bmatrix} + \begin{bmatrix} -2 & 3 \\ 1 & -1 \\ 2 & -2 \end{bmatrix}$$

Now You Try It!

Example

Find the indicated operations

(a)

$$\begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 0 & -2 \end{bmatrix}$$

(b)

$$\begin{bmatrix} -3 & 2 \\ 4 & -1 \end{bmatrix} - \begin{bmatrix} -2 & 5 \\ -1 & 3 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 & 3 & -2 \end{bmatrix}$$

Scalar Multiplication

If k is a number and M is a matrix, we can form the scalar product kM by just multiplying every element of M by k.

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Example

Find

$$-2\begin{bmatrix} 3 & -1 & 0 \\ -2 & 1 & 3 \\ 0 & -1 & -2 \end{bmatrix}$$

Now You Try It

Example

Find

$$5\begin{bmatrix} 1 & -1 \\ 0 & -2 \\ 2 & -3 \\ 3 & 3 \end{bmatrix}$$

Matrix Multiplication

In order to define matrix multiplication, it is easier to first define the product of a row matrix with a column matrix.

Matrix Multiplication

Definition

Suppose we have a $1 \times n$ *row matrix* A *and an* $n \times 1$ *column matrix* B *where*

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

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Then

$$AB = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

It is very important that the number of columns in A *matches the number of rows in* B.

Matrix Multiplication

Example

Find

$$\begin{bmatrix} -1 & 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \\ -1 \end{bmatrix}$$

Now You Try It!

Example

Find

$$\begin{bmatrix} 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

Matrix Multiplication

Definition (Matrix Multiplication)

Let A be an $m \times p$ matrix and let B be a $p \times n$ matrix. Let R_i denote the matrix formed by the i^{th} row of A and let C_j denote the matrix formed by the j^{th} column of B. Then the ij^{th} element of the matrix product AB is $R_i C_j$.

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Remark

It is very important that the number of columns of A matches the number of rows of B, otherwise the products $R_i C_j$ would not be able to be defined. That is, if A is an $m \times n$ matrix and B is an $p \times q$ matrix, the product AB is defined if and only if $n = p$.

Matrix Multiplication

Let's find the matrix product

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 3 & 5 & 7 \end{bmatrix}$$

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$$= \begin{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} & \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} & \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix} \\ \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} & \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} & \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix} \end{bmatrix}$$

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$$= \begin{bmatrix} 1 \cdot 1 + 2 \cdot 3 & 1 \cdot 2 + 2 \cdot 5 & 1 \cdot 4 + 2 \cdot 7 \\ 2 \cdot 1 + 1 \cdot 3 & 2 \cdot 2 + 1 \cdot 5 & 2 \cdot 4 + 1 \cdot 7 \end{bmatrix}$$

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$$= \begin{bmatrix} 1 + 6 & 2 + 10 & 4 + 14 \\ 2 + 3 & 4 + 5 & 8 + 7 \end{bmatrix}$$

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$$= \begin{bmatrix} 1 + 6 & 2 + 10 & 4 + 14 \\ 2 + 3 & 4 + 5 & 8 + 7 \end{bmatrix} = \begin{bmatrix} 7 & 12 & 18 \\ 5 & 9 & 15 \end{bmatrix}$$

Matrix Multiplication

Example

Let $A = \begin{bmatrix} -1 & 0 & 3 & -2 \\ 1 & 2 & 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ 2 & 3 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$, $D = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}$. *Find the following products, if possible.*

(a) AB

(b) BA

(c) CD

Matrix Multiplication

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Let $A = \begin{bmatrix} -1 & 0 & 3 & -2 \\ 1 & 2 & 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ 2 & 3 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$, $D = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}$. *Find the following products, if possible.*

(a) AB

(b) BA

(c) CD

(d) DC

(e) CB

(f) D^2

Interesting Fact!

Solution

(d) $\begin{bmatrix} -6 & -12 \\ 3 & 6 \end{bmatrix}$

(e) *Not defined.*

(f) $\begin{bmatrix} 8 & -16 \\ -4 & 8 \end{bmatrix}$

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(d) $\begin{bmatrix} -6 & -12 \\ 3 & 6 \end{bmatrix}$

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Remark

Note that parts (c) and (d) show that matrix multiplication is not commutative. That is, it is not necessarily true that $AB = BA$ for matrices A and B, even if both matrix products are defined.